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JANUARY, 1860.

EDITED BY

J. D. RUNKLE, A.M., A.A.S.

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THE

MATHEMATICAL MONTHLY.

Vol. II. . . . JANUARY, 1860. . . . No. IV.

PRIZE PROBLEMS FOR STUDENTS.

I. A and B can do a piece of work in m days; B and C in n days; in what time can A and C do the same, it being supposed that A can do p times as much as B in a given time?

II. Show that $\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{(2+\sqrt{3})}} + \frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{(2-\sqrt{3})}} = \sqrt{2}$.

III. Find the roots of the equation $x^3 - 6x = 4$, by trigonometry.

IV. Four persons, A, B, C, D, in order, beginning with A, cut a pack of cards, replacing them after each cut, on condition that the first who cuts a heart shall win. What are their respective probabilities of success?

V. The notation of Problem V. in the November number of the Monthly being retained, prove that in the plane

$$\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C;$$

and in the sphere

$$\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C\,\frac{\cos(\rho+\delta)\cos(\rho-\delta)}{\cos^2 r\cos^2\rho},$$

$$\cot\tfrac{1}{2}A\,\frac{\cos\delta'}{\cos\rho'} + \cot\tfrac{1}{2}B\,\frac{\cos\delta''}{\cos\rho''} + \cot\tfrac{1}{2}C\,\frac{\cos\delta'''}{\cos\rho'''} = \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C\,\frac{\cos\delta}{\cos\rho}.$$

The solutions of these problems must be received by March 1, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. I., Vol. II.

The first Prize is awarded to J. D. Van Buren, Rensselaer Polytechnic Institute, Troy, N. Y.

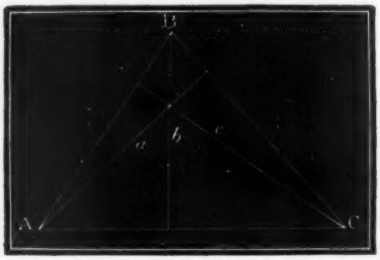
The second Prize is awarded to O. B. Wheeler, Sophomore class, University of Michigan, at Ann Arbor.

The third Prize is awarded to George H. Tower, Classical High School, Petersham, Mass.

Prize Solution of Problem II.

By Perrin B. Page, Nunda, N. Y.

In any plane triangle, prove that the sines of the angles are inversely as the perpendiculars let fall from them upon the opposite sides.



Let $A\,B\,C$ be the triangle, and denote by $a, b, c,$ the perpendiculars dropped respectively from the angles $A, B, C.$ By trigonometry

(1.) $$\frac{\sin A}{B\,C} = \frac{\sin B}{A\,C} = \frac{\sin C}{A\,B}.$$

By geometry, twice the area of $A\,B\,C$ is

(2.) $$B\,C \times a = A\,C \times b = A\,B \times c.$$

Multiplying equations (1) and (2) together, member by member, gives,

$$\sin A \times a = \sin B \times b = \sin C \times c;\ \text{or},$$
$$\sin A : \sin B : \sin C = c : b : a.$$

Second Solution. — By definition,

$$\sin A = \frac{b}{A\,B} = \frac{c}{A\,C}, \qquad \sin B = \frac{a}{A\,B} = \frac{c}{B\,C},$$

$$\therefore \frac{\sin A}{\sin B} = \frac{b}{a} = \frac{B\,C}{A\,C}.$$

$$\therefore \sin A : \sin B = b : a = BC : AC,$$

which not only proves the proposition, but also that *the sines of the angles are proportional to their opposite sides.* This is a slight modification of W. C. HENCK's solution.

PRIZE SOLUTION OF PROBLEM III.

By ISAAC H. TURRELL, Mt. Carmel, Indiana.

Having given the diagonals of a quadrilateral inscribed in a given circle to determine its sides geometrically, when the diagonals intersect each other at right angles. — Communicated by Professor D. W. HOYT.

Let AB and CD be the given diagonals. Inscribe AB in the given circle. Next, draw parallel to AB and at a distance from the centre equal to one half the other diagonal CD, the line CC'. From either extremity of this line, as C, draw a line perpendicular to AB and produce it till it meets the circumference in D. The line CD is therefore inscribed in the circle, perpendicular to AB, and $ACBD$ is the required quadrilateral. CHARLES B. BOUTELLE's solution is essentially the same as the above.

PRIZE SOLUTION OF PROBLEM IV.

By J. D. VAN BUREN, Rensselaer Polytechnic Inst., Troy, N. Y.

Given

$$(1.)\quad xw + yz = ab, \qquad (2.)\quad xy + zw = ad$$

$$(3.)\quad xz + yw = bd, \qquad (4.)\quad x^2 + w^2 = y^2 + z^2;$$

to find the values of x, y, z, and w. — Communicated by Professor D. W. HOYT.

Adding and subtracting (1) and (3), (1) and (2), and also multiplying (3) and (2) together, we obtain

$$(5.)\quad x + y = \frac{bd + ab}{z + w}, \qquad (7.)\quad x + z = \frac{ab + ad}{w + y},$$

$$(6.)\quad x - y = \frac{bd - ab}{z - w}, \qquad (8.)\quad x - z = \frac{ab - ad}{w - y},$$

$$(9.)\qquad yz\,(x^2 + w^2) + xw\,(z^2 + y^2) = a\,b\,d^2.$$

Combining the products of (5) and (6), and (7) and (8), with (9) and (4) respectively, we get

$$x^2 - y^2 = \pm\, b\sqrt{d^2 - a^2},\; y^2 - w^2 = \pm\, a\sqrt{b^2 - d^2},\; x^2 + w^2 = d^2 = y^2 + z^2.$$

Therefore,

$$2\,x^2 = d^2 \pm b\,\sqrt{d^2 - a^2} \pm a\,\sqrt{b^2 - d^2},$$
$$2\,y^2 = d^2 \mp b\,\sqrt{d^2 - a^2} \pm a\,\sqrt{b^2 - d^2},$$
$$2\,w^2 = d^2 \mp b\,\sqrt{d^2 - a^2} \mp a\,\sqrt{b^2 - d^2},$$
$$2\,z^2 = d^2 \pm b\,\sqrt{d^2 - a^2} \mp a\,\sqrt{b^2 - a^2}.$$

Whence we obtain the values of the required quantities by dividing by 2 and extracting the square root. Similar solutions were given by William W. Johnson and O. B. Wheeler.

O. B. Wheeler remarks that the equations of Problem IV. are readily obtained from the figure of Problem III. From geometry we have, "*The rectangle of the two diagonals is equivalent to the sum of the rectangles of the opposite sides*," which is (1). We have (2) and (3) by the proposition, "*In any triangle the rectangle contained by two sides is equivalent to the rectangle contained by the diameter of the circumscribed circle and the perpendicular let fall on the third side;*" and (4) from the sum of the equations

$$A\,E^2 = w^2 - D\,E^2 = y^2 - E\,C^2,$$
$$B\,E^2 = x^2 - E\,C^2 = z^2 - D\,E^2.$$

The connection between Problems III. and IV. was also noticed by W. C. Cleveland, who gave the following geometrical demonstration of (4).

Let the chord $C\,C'$ be drawn parallel to $A\,B$; then $A\,C' = B\,C = x$, angle $C'\,C\,D = 90°$;

$$\therefore\; C'\,D \text{ is a diameter;} \qquad \therefore\; x^2 + w^2 = d^2.$$

Similarly $y^2 + z^2 = d^2$; whence equation (4).

Prize Solution of Problem V.

By J. D. Van Buren, Rensselaer Polytechnic Institute, Troy, N. Y.

If, in a plane or spherical triangle, A, B, C denote the angles, and a, b, c the opposite sides respectively; if r, ρ denote the radii of the circumscribed and inscribed circles, and δ the distance between the centres of these circles; then in the plane triangle

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C,$$

and in the spherical triangle

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C \frac{\cos \delta}{\cos r \cos \rho}.$$

Plane Triangle. We have from plane trigonometry,

$$\sin A = \frac{2k}{bc}, \quad \sin B = \frac{2k}{ac}, \quad \sin C = \frac{2k}{ab},$$

when $k = \sqrt{s(s-a)(s-b)(s-c)}$, and $s = \frac{1}{2}(a+b+c)$.

$$\therefore \sin A + \sin B + \sin C = \frac{2k(a+b+c)}{abc} = \frac{4ks}{abc}.$$

Also, $\cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}$, $\cos \frac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}}$, $\cos \frac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}$.

$$\therefore \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C = \frac{ks}{abc},$$

$$\therefore \sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C.$$

Spherical Triangle. We have, from spherical trigonometry,

$$\sin A = \frac{2k}{\sin b \sin c}, \quad \sin B = \frac{2k}{\sin a \sin c}, \quad \sin C = \frac{2k}{\sin a \sin b},$$

in which

$k = \sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)}$, and $s = \frac{1}{2}(a+b+c)$.

(1.) $$\therefore \sin A + \sin B + \sin C = \frac{2k(\sin a + \sin b + \sin c)}{\sin a \sin b \sin c}.$$

Also,

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}, \quad \cos \tfrac{1}{2} B = \sqrt{\frac{\sin s \sin(s-b)}{\sin a \sin c}},$$

$$\cos \tfrac{1}{2} C = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}.$$

$$\therefore \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C = \frac{k\sin s}{\sin a\sin b\sin c}.$$

But we have (CHAUVENET's Trig., p. 251, and Eq. 319),

$$\frac{\cos\delta}{\cos r\sin\varrho} = \frac{\sin a + \sin b + \sin c}{2k}, \quad \tan\varrho = \frac{k}{\sin s},$$

$$\therefore \sin\varrho = \frac{k\cos\varrho}{\sin s} \qquad \therefore \frac{\cos\delta}{\cos r\cos\varrho} = \frac{\sin a + \sin b + \sin c}{2\sin s}.$$

$$(2.) \quad \therefore \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\,.\,\frac{\cos\delta}{\cos r\cos\varrho} = \frac{k(\sin a + \sin b + \sin c)}{2\sin a\sin b\sin c}.$$

Therefore, by comparing (1) and (2) we get

$$\sin A + \sin B + \sin C = 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\,.\,\frac{\cos\delta}{\cos r\cos\varrho}.$$

SECOND SOLUTION.

By O. B. WHEELER, University of Michigan.

Plane Triangle. Since in a plane triangle $\frac{1}{2}(A + B + C) = \frac{1}{2}\pi$, we have

(1.) $\cos\frac{1}{2}(A + B + C) = \cos\frac{1}{2}\pi = 0.$

(2.) $\cos\frac{1}{2}(-A + B + C) = \cos(\frac{1}{2}\pi - A) = \sin A,$

(3.) $\cos\frac{1}{2}(A - B + C) = \cos(\frac{1}{2}\pi - B) = \sin B,$

(4.) $\cos\frac{1}{2}(A + B - C) = \cos(\frac{1}{2}\pi - C) = \sin C.$

Taking the sum of (1) and (2), and also of (3) and (4), we obtain

(5.) $2\cos\frac{1}{2}A\cos\frac{1}{2}(B + C) = \sin A,$

(6.) $2\cos\frac{1}{2}A\cos\frac{1}{2}(B - C) = \sin B + \sin C.$

Adding (5) and (6), developing and reducing, we have

$$4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C = \sin A + \sin B + \sin C.$$

MR. WHEELER's solution for the spherical triangle is essentially the same as the one already given.

SIMON NEWCOMB.
W. P. G. BARTLETT.
TRUMAN HENRY SAFFORD.

ANOTHER SOLUTION OF PRIZE PROBLEM I., No. IX.

By George Eastwood, Saxonville, Mass.

There are two ways of solving a geometrical problem *geometrically*. One way is to solve it by *synthesis*, the other way is to solve it by *analysis*. In solving a problem by analysis, we assume that the work *is done;* and then proceed, as the word implies, to take it to pieces, and to examine and develop the properties of each part and its relations to all the other parts. To solve the same problem by synthesis, we proceed to construct it upon elementary principles, from the given data and such combinations of them, as the exigencies of the case may require. We have a beautiful illustration of the synthetic method in Mr. Evans's solution of Prize Problem I. of No. IX., published in the last Monthly. The following solution of the same problem is offered to young students as a simple specimen of the analytic method.*

Analysis. Suppose the thing done, that $A\,C\,B$ is the required triangle, and that $A\,F\,B\,E$, $G\,H\,I$, are its circumscribing and inscribed circles. Join O, O', the centres of the circles, and draw the radii $O'\,G$, $O'\,H$, $O'\,I$, to the points of contact G, H, I. Draw the diameter $E\,F$ perpendicular to the hypothenuse $A\,B$, and $C\,D$ perpendicular to $E\,F$.

By the question, we have given,

$$A\,C - B\,C = (A\,G + G\,C) - (B\,H + H\,C)$$
$$= A\,I - B\,I = 2\;O\,I = 2\;M\,N \text{ suppose.}$$

---

* Strictly speaking, the two ways of solving a geometrical problem, above indicated, are not two distinct and independent methods of doing the same thing, but rather the different parts of one full and perfect method. The one process is the reverse of the other, and the use of each is indispensable to a complete solution.

Also, we have given,

$$AB - BC = (AI + IB) - (IB + IO') = AI - IO',$$

which suppose equal to $2\,LK$. Take $2\,MN$ from $4\,LK$, then

$$AI + BI - 2\,IO' = 4\,LK - 2\,MN;$$

$$\therefore\ AO = 2\,LK - MN + IO'.$$

By Prop. VI., *Liverpool Student*, $EO \,.\, FD = OI^2 = MN^2$; so that when $EO = AO$ is found, FD will be given.

By Prop. 47, Euc. I.

$$OI^2 + O'I^2 = O'O^2 = AO(AO - 2\,O'I)$$
$$= (2\,LK - MN)^2 - O'I^2 = (2\,LK - OI)^2 - O'I^2,$$
$$\therefore\ O'I^2 = 2\,LK(LK - MN).$$

As LK and MN are both given lines, therefore $O'I$ is given. Hence this

Construction. Find (Euc. VI. 13) $O'I$ a mean proportional between $2\,LK$ and $LK - MN$, and with centre O and radius $OA = 2\,LK - MN + O'I$, describe a circle. Draw the horizontal and vertical diameters AB, FE, and upon FE apply (Euc. VI. 11) FD a third proportional to EO, OI. Draw DC parallel to AB to meet the circle in C, and join AC, BC; then ACB is the required triangle. The demonstration is evident from the analysis.

NOTES AND QUERIES.

1. *Note on Decimals.* — Decimals should be taught in written arithmetic in connection with whole numbers. Let them be treated in the same manner as whole numbers, and not as common fractions, there being no necessity for confusing the mind of the pupil by writing the denominator. In 31.3 there is no more need of indicating that the three standing in tenths' place is divided by ten, than

that the three standing in ten's place is multiplied by ten. The pupil can readily be made to understand these relations without the divisor or multiplier. There should be no separate divisions in arithmetic for Decimal Fractions and Federal Money. The four simple rules should embrace these, and Percentage and Interest should be treated immediately after as a development of the decimal notation. The tables and Compound Numbers should immediately precede Vulgar Fractions, — the former *in*creasing in an irregular ratio, and the latter *de*creasing in an irregular ratio. If authors in writing arithmetics would observe this order, it would result in great advantage to the learner. — SAMUEL P. BATES, Superintendent of Public Instruction, Crawford County, Pa.

2. *Reduction of Fractions to a Common Denominator.* — After the pupil has thoroughly learned that multiplying or dividing both terms of a fraction by the same number does not change its value, he is then prepared to learn how to reduce fractions to a common denominator. The following process is very simple, and the pupil should be required to repeat it until it is thoroughly understood.

Let it be required, for example, to reduce $\frac{7}{14}$, $\frac{9}{15}$, $\frac{23}{28}$, $\frac{19}{35}$, $\frac{12}{21}$ to a common denominator. First, resolve the terms of all the fractions into their prime factors, and write them in a horizontal row. Now it is evident that when the denominators are common they will contain the same factors. Let us first compare them with reference to the factor 7. This factor is found in all the denominators but one; and in order

1st.	2d.	3d.	4th.	5th.
$\frac{7}{2\cdot7}$	$\frac{3\cdot3}{3\cdot5}$	$\frac{23}{2\cdot2\cdot7}$	$\frac{19}{5\cdot7}$	$\frac{2\cdot2\cdot3}{3\cdot7}$
$\frac{7}{2\cdot7}$	$\frac{3\cdot3\cdot7}{3\cdot5\cdot7}$	$\frac{23}{2\cdot2\cdot7}$	$\frac{19}{5\cdot7}$	$\frac{2\cdot2\cdot3}{3\cdot7}$
$\frac{7}{2\cdot7}$	$\frac{3\cdot7}{5\cdot7}$	$\frac{23}{2\cdot2\cdot7}$	$\frac{19}{5\cdot7}$	$\frac{2\cdot2}{7}$
$\frac{7\cdot5}{2\cdot7\cdot5}$	$\frac{3\cdot7}{5\cdot7}$	$\frac{23\cdot5}{2\cdot2\cdot7\cdot5}$	$\frac{19}{5\cdot7}$	$\frac{2\cdot2\cdot5}{7\cdot5}$
$\frac{2\cdot7\cdot5}{2\cdot2\cdot7\cdot5}$	$\frac{2\cdot2\cdot3\cdot7}{2\cdot2\cdot5\cdot7}$	$\frac{23\cdot5}{2\cdot2\cdot7\cdot5}$	$\frac{2\cdot2\cdot19}{2\cdot2\cdot5\cdot7}$	$\frac{2\cdot2\cdot2\cdot2\cdot5}{2\cdot2\cdot7\cdot5}$

to make them common with reference to 7, it must either be introduced into the denominator of the 2d fraction, or removed from the denominators of all the other fractions. But to remove a factor from the denominator of a fraction without changing its value, it must be removed from the numerator also. Now 7 is a factor in only one of the numerators, and the denominators cannot be made common by removing the 7. It must therefore be introduced into the denominator of the second fraction to make the denominators common; and also into the numerator, in order not to change the value of the fraction. We thus get the second row. Next compare the denominators with reference to the factor 3. It is found in only two of them, from which it may be removed, since it is also found in the corresponding numerators. Thus we get the third row. (It is plain, that instead of removing the factor 3 from the 2d and 5th fractions, the denominators might be made common with reference to 3 by introducing it into the terms of all the other fractions. We should not, however, in this way reduce the fractions to their least common denominator.) Next introduce the factor 5 and get the fourth row; and lastly, introduce the factors 2.2, which makes all the denominators common. — TEACHER.

3. *Note on the superior limit of the Roots of an Equation.*

PROP. I. *The greatest negative coefficient of an equation, plus unity, is a superior limit of its roots.*

PROOF. Let us assume the general equation,

$$x^n \pm Ax^{n-1} \pm Bx^{n-2} \ldots\ldots \pm Nx^{n-r+1} \pm V = 0.$$

A superior limit of the roots of an equation must produce a positive result when substituted for x; that is, the sum of all the positive terms must exceed the sum of all the negative ones by some quantity R. The most unfavorable case evidently is that in which *all the terms after the first are negative and have equal coefficients.* Under this assumption our equation becomes

$$x^n - Nx^{n-1} - Nx^{n-2} \ldots\ldots - N = 0.$$

Suppose a to be greater than the greatest root, then

$$a^n - Na^{n-1} - Na^{n-2} \ldots\ldots - N = R; \text{ or}$$

$$a^n - N(a^{n-1} - a^{n-2} \ldots\ldots - 1) = R; \text{ or}$$

$$a^n - N\left(\frac{a^n - 1}{a - 1}\right) = R.$$

$$\therefore\ a^n = N\left(\frac{a^n - 1}{a - 1}\right) + R = \frac{Na^n}{a - 1} - \frac{N}{a - 1} + R.$$

Now suppose a to have been so taken that $\frac{N}{a - 1} = R$, as may always be done; then we shall have

$$a^n = \frac{Na^n}{a - 1}, \qquad \text{or } a - 1 = N, \qquad \text{or } a = N + 1.$$

But $a > x$, $\therefore N + 1 > x$; and hence $N + 1$ is a superior limit, as was to be proved.

Prop. 2. *If unity be added to that root of the greatest negative coefficient, which is denoted by the number of terms preceding the first negative term, the result will be a superior limit of the roots of the equation.*

In the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots - Kx^{n-r} \pm \ldots - Px^{n-r-s} \ldots \pm V = 0,$$

suppose K to be the *first*, and P the *greatest*, negative coefficient, and r the number of terms which precede the first negative term. Evidently the most unfavorable case is that in which the coefficients A, B, C, &c., preceding K are zero, and all the coefficients K, L, M, &c., are all negative and equal. Under this supposition our equation becomes

$$x^n - Px^{n-r} - Px^{n-r-1} \ldots\ldots - P = 0.$$

Suppose a to be greater than the greatest root, then

$$a^n - Pa^{n-r} - Pa^{n-r-1} \ldots\ldots - P = R; \text{ or}$$

$$a^n - P\left(\frac{a^{n-r+1} - 1}{a - 1}\right) = R,$$

$$\therefore a^n = P\left(\frac{a^{n-r+1}-1}{a-1}\right) + R = \frac{Pa^{n-r+1}}{a-1} - \frac{P}{a-1} + R.$$

Suppose a to have been so taken that $\frac{P}{a-1} = R$, as may always be done; then we shall have

$$a^n = \frac{Pa^{n-r+1}}{a-1}, \text{ or } (a-1)\,a^{r-1} = P. \quad \text{But } a-1 < a;$$

$$\therefore (a-1)^{r-1} < a^{r-1},\ (a-1)^r < (a-1)\,a^{r-1}.$$

Therefore, since $(a-1)\,a^{r-1} = P$, $(a-1)^r < P$;

$$\therefore a - 1 < \sqrt[r]{P}, \text{ or } a < \sqrt[r]{P} + 1.$$

But $a > x$, $\therefore \sqrt[r]{P} + 1 > x$, as was to be proved.— E. B. Weaver, Normal School, Lancaster Co., Pa.

ANALYTICAL SOLUTIONS "OF THE TEN PROBLEMS IN THE TANGENCIES OF CIRCLES; AND ALSO OF THE FIFTEEN PROBLEMS IN THE TANGENCIES OF SPHERES."

By George W. Coaklay, Professor of Mathematics, College of St. James, Washington Co., Md.

Part of the title of this article is derived from that of Major Alvord's paper in the eighth volume of the "*Smithsonian Contributions to Knowledge.*" The analytic solutions here given were obtained in the winter of 1856, soon after I became acquainted with Major Alvord's paper; they were attempted in consequence of the statements in that paper, that no mathematician had yet succeeded in reducing the analytical solution of these problems to equations of the second degree. At the end of his geometrical solutions, Major Alvord states, that the algebraic solution leads to an equation of the *eighth degree* when applied to the most general problem of the tangencies of circles; and to an equation of the *sixteenth degree* when applied to the most general problem of the tangencies of spheres. It will be found, however, that in this article both of these general

problems, which include all the others contained in the Smithsonian paper, are solved completely by equations not surpassing the second degree. The solution of the *particular* problems included in the two *general* problems here treated may be obtained either by applying to them the general equations developed in this article; or by treating them separately in a similar manner, the process being of still easier application to most of these particular problems. Thus the whole twenty-five problems of the Smithsonian paper may be considered as reduced to equations of the second degree, when the two most comprehensive problems have been so reduced.

First General Problem.

"To draw a Circle tangent to three given Circles."

Let the radii of the given circles be denoted by r_1, r_2, r_3, respectively, and let the corresponding co-ordinates of their centres be denoted by (a_1, b_1), (a_2, b_2), (a_3, b_3); also let ϱ, (α, β) be the radius and centre of the required circle, all being referred to any system of rectangular co-ordinate axes in the common plane of the circles. The equations of these circles will be respectively

$$(1.)\qquad (x-a_1)^2+(y-b_1)^2=r_1^2,$$

$$(2.)\qquad (x-a_2)^2+(y-b_2)^2=r_2^2,$$

$$(3.)\qquad (x-a_3)^2+(y-b_3)^2=r_3^2,$$

$$(4.)\qquad (x-\alpha)^2+(y-\beta)^2=\varrho^2,$$

Let x_1, y_1 be the point of contact of (1) and (4),
" x_2, y_2 " " " (2) and (4),
" x_3, y_3 " " " (3) and (4).

Then the point x_1, y_1 must satisfy the conditions,

$$(5.)\qquad (x_1-a_1)^2+(y_1-b_1)^2=r_1^2,$$

$$(6.)\qquad (x_1-\alpha)^2+(y_1-\beta)^2=\varrho^2,$$

(7.) $$\frac{dy_1}{dx_1} = -\frac{x_1 - a_1}{y_1 - b_1} = -\frac{x_1 - \alpha}{y_1 - \beta},$$

and similar conditions exist for the circles (2) and (3) combined with (4); but they may readily be derived from (5), (6), and (7) by simply changing the subscript 1 into 2 or 3.

From (7) we have

$$\frac{(x_1 - a_1)^2}{(y_1 - b_1)^2} + 1 = \frac{(x_1 - \alpha)^2}{(y_1 - \beta)^2} + 1,$$

$$\therefore \frac{r_1^2}{(y_1 - b_1)^2} = \frac{\varrho^2}{(y_1 - \beta)^2}.$$

(8.) $$\therefore y_1 - \beta = \pm \frac{\varrho}{r_1}(y_1 - b_1),$$

(9.) $$x_1 - \alpha = \pm \frac{\varrho}{r_1}(x_1 - a_1),$$

$$\therefore y_1 = \frac{r_1 \beta \mp b_1 \varrho}{r_1 \mp \varrho}, \qquad x_1 = \frac{r_1 \alpha \mp a_1 \varrho}{r_1 \mp \varrho},$$

$$\therefore y_1 - b_1 = \frac{r_1(\beta - b_1)}{r_1 \mp \varrho}, \qquad x_1 - a_1 = \frac{r_1(\alpha - a_1)}{r_1 \mp \varrho}.$$

Hence equation (5) becomes

$$(\alpha - a_1)^2 + (\beta - b_1)^2 = (r_1 \mp \varrho)^2,$$

(A.) and in like manner $(\alpha - a_2)^2 + (\beta - b_2)^2 = (r_2 \mp \varrho)^2,$

$$(\alpha - a_3)^2 + (\beta - b_3)^2 = (r_3 \mp \varrho)^2.$$

These equations evidently express the fact that the distance between the centres of any two of the tangent circles is equal to the sum or difference of their radii according as the contact is external or internal. It is also evident that the following are all the cases that can possibly occur, viz.: —

The required circle touches the circles (1), (2), (3) externally,
" " " " (1), (2), (3) internally,
" " " " (1), (2), ext. and (3) int.,
" " " " (1), (3), ext. and (2) int.,

The required circle touches the circles (2), (3), ext. and (1) int.,
" " " " (1), (2), int. and (3) ext.,
" " " " (1), (3), int. and (2) ext.,
" " " " (2), (3), int. and (1) ext.,
making eight cases, (the number found by Major ALVORD,) each of which is solved by a proper combination of the algebraic signs in the equations (A). In order to embrace all the cases in one solution, let $R_1 = \pm r_1$, $R_2 = \pm r_2$, $R_3 = \pm r_3$; then equations (A) may be written

$$\text{(A}'.) \quad \begin{aligned} (\alpha - a_1)^2 + (\beta - b_1)^2 &= \{\pm (\varrho - R_1)\}^2 \\ (\alpha - a_2)^2 + (\beta - b_2)^2 &= \{\pm (\varrho - R_2)\}^2 \\ (\alpha - a_3)^2 + (\beta - b_3)^2 &= \{\pm (\varrho - R_3)\}^2. \end{aligned}$$

To find the values of the unknown quantities, α, β, ϱ, let

$$(10.) \qquad \alpha = A_1\, \varrho + A_2, \qquad \beta = B_1\, \varrho + B_2,$$

A_1, A_2, B_1, B_2 being functions of the *given* quantities, readily determined in each case by equations of the *first degree* in the following manner.

Developing the squares in the three equations (A′), and subtracting in turn the members of the first from those of the second and third, we have

$$(11.) \quad \begin{aligned} 2(a_1-a_2)\,\alpha+2(b_1-b_2)\,\beta &= 2(R_1-R_2)\,\varrho+a_1^2-a_2^2+b_1^2-b_2^2-(R_1^2-R_2^2), \\ 2(a_1-a_3)\,\alpha+2(b_1-b_3)\,\beta &= 2(R_1-R_3)\,\varrho+a_1^2-a_3^2+b_1^2-b_3^2-(R_1^2-R_3^2). \end{aligned}$$

Substituting from (10), the values of α and β, equations (11) become

$$2\,\{(a_1 - a_2)\,A_1 + (b_1 - b_2)\,B_1\}\,\varrho + 2\,(a_1 - a_2)\,A_2 + 2\,(b_1 - b_2)\,B_2 = \\ 2\,(R_1 - R_2)\,\varrho + a_1^2 - a_2^2 + b_1^2 - b_2^2 - (R_1^2 - R_2^2),$$

$$2\,\{(a_1 - a_3)\,A_1 + (b_1 - b_3)\,B_1\}\,\varrho + 2\,(a_1 - a_3)\,A_2 + 2\,(b_1 - b_3)\,B_2 = \\ 2\,(R_1 - R_3)\,\varrho + a_1^2 - a_3^2 + b_1^2 - b_3^2 - (R_1^2 - R_3^2).$$

Now the two equations (11) are insufficient to determine the values of the three unknown quantities, α, β, ϱ; hence the result of the

preceding substitutions must be to give *identical equations*, or those capable of being verified by any value of ϱ whatever; hence the coefficients of the corresponding powers of ϱ must be equal.

Therefore

$$(12.)\quad \begin{aligned} (a_1 - a_2)\,A_1 + (b_1 - b_2)\,B_1 &= R_1 - R_2 \\ (a_1 - a_3)\,A_1 + (b_1 - b_3)\,B_1 &= R_1 - R_3, \end{aligned}$$

$$(13.)\quad \begin{aligned} (a_1 - a_2)A_2 + (b_1 - b_2)B_2 &= \tfrac{1}{2}\{a_1^2 - a_2^2 + b_1^2 - b_2^2 - (R_1^2 - R_2^2)\}, \\ (a_1 - a_3)A_2 + (b_1 - b_3)B_2 &= \tfrac{1}{2}\{a_1^2 - a_3^2 + b_1^2 - b_3^2 - (R_1^2 - R_3^2)\}. \end{aligned}$$

Hence

$$(14.)\quad \begin{aligned} A_1 &= \frac{(R_1 - R_2)\,(b_1 - b_3) - (R_1 - R_3)\,(b_1 - b_2)}{(a_1 - a_2)\,(b_1 - b_3) - (a_1 - a_3)\,(b_1 - b_2)} \\ B_1 &= \frac{(R_1 - R_3)\,(a_1 - a_2) - R_1 - R_2)\,(a_1 - a_3)}{(a_1 - a_2)\,(b_1 - b_3) - (a_1 - a_3)\,(b_1 - b_2)}, \end{aligned}$$

and by putting

$$\begin{aligned} D &= a_1^2 - a_2^2 + b_1^2 - b_2^2 - (R_1^2 - R_2^2) \\ E &= a_1^2 - a_3^2 + b_1^2 - b_3^2 - (R_1^2 - R_3^2) \\ F &= (a_1 - a_2)\,(b_1 - b_3) - (a_1 - a_3)\,(b_1 - b_2), \end{aligned}$$

we also get

$$(15.)\quad A_2 = \frac{(b_1 - b_3)\,D - (b_1 - b_2)\,E}{2\,F}, \quad B_2 = \frac{(a_1 - a_2)\,E - (a_1 - a_3)\,D}{2\,F}.$$

Thus the coefficients, A_1, B_1, A_2, B_2, of equations (10) are entirely determined in terms of the given quantities. From equations (10) we have

$$(16.)\quad \begin{aligned} \alpha - a_1 &= A_1(\varrho - R_1) + A_2 + A_1 R_1 - a_1 = A_1(\varrho - R_1) + m \\ \beta - b_1 &= B_1(\varrho - R_1) + B_2 + B_1 R_1 - b_1 = B_1(\varrho - R_1) + n, \end{aligned}$$

by making

$$(17.)\quad m = A_2 + A_1 R_1 - a_1, \qquad n = R_2 + B_1 R_1 - b_1.$$

Hence the first equation of (A′) becomes

$$(18.)\quad (\varrho - R_1)^2 - \frac{2\,(m\,A_1 + n\,B_1)}{1 - (A_1^2 + B_1^2)}\,(\varrho - R_1) = \frac{m^2 + n^2}{1 - (A_1^2 + B_1^2)}.$$

Hence

(19.) $$\varrho = R_1 + \frac{m A_1 + n B_1 \pm \sqrt{m^2 + n^2 - (n A_1 - m B_1)^2}}{1 - (A_1^2 + B_1^2)}.$$

Thus the equations (10), (14), (15), (17), and (19) completely solve the problem for every possible case.

It is evident that the problem is impossible whenever

$$(n A_1 - m B_1)^2 > m^2 + n^2;$$

and that of the two roots in (19), the one which gives a negative value for ϱ is to be interpreted as giving a circle of contact the reverse of that implied by the values assumed for R_1, R_2, and R_3. If, for example, the value assumed for R_2 implied *external* contact, then the negative value of ϱ would indicate *internal* contact with the second given circle.

I shall apply the preceding formulæ to determine the circle which passes through three given points, regarded as circles of infinitely small radius. Let the axis of abscissas pass through two of the given points, (a_2, b_2) and (a_3, b_3), and let the origin bisect the distance between them; hence $a_3 = -a_2$, $b_2 = b_3 = 0$; also, $r_1 = r_2 = r_3 = 0 = R_1 = R_2 = R_3$.

Hence $$A_1 = 0, \quad B_1 = 0, \quad A_2 = 0,$$

$$B_2 = \tfrac{1}{2}\frac{a_1^2 + b_1^2 - a_2^2}{b_1}; \quad m = -a_1, \quad n = \frac{a_1^2 - b_1^2 - a_2^2}{2 b_1},$$

$$\varrho = \pm\sqrt{m^2 + n^2} = \pm\sqrt{\frac{4 a_1^2 b_1^2 + (a_1^2 - b_1^2 - a_2^2)^2}{4 b_1^2}};$$

$$\alpha = 0, \beta = \tfrac{1}{2}\,\frac{a_1^2 - b_1^2 - a_2^2}{b_1}.$$

The remaining point, (a_1, b_1) is thus far subject to no restriction; it is anywhere on the plane of the co-ordinate axes. If now we suppose first that it is on the same straight line with the other two points, then $b_1 = 0$; hence $\beta = \infty$, $\varrho = \pm\infty$. Hence the radius

of the circle passing through three given points on the same straight line is infinite, and its centre is at an infinite distance from that line. The double sign of ρ shows that the contact may be considered in this case either external or internal. Secondly, if we suppose the point (a_1, b_1) to be on the axis of ordinates, at a distance from the origin equal to a_2, then $a_1 = 0$, $b_1 = a_2$, $\beta = 0$, $\alpha = 0$, $m = 0$, $n = -a_2$, $\rho = \pm a_2$, the circle having either external or internal contact with the three given points; and it is obvious that the results are correct.

As a second example, let it be required to find the circle which shall touch *externally* the three circles whose radii and centres are as follows:

$$\begin{array}{llll} r_1 = 1, & r_2 = 3, & r_3 = 2, & \text{hence } R_1 = -r_1 = -1, \\ a_1 = 0, & a_2 = 5, & a_3 = -3, & \quad " \quad R_2 = -r_2 = -3, \\ b_1 = 0, & b_2 = 0, & b_3 = 4, & \quad " \quad R_3 = -r_3 = -2. \end{array}$$

It will be found that $A_1 = -0.4$, $B_1 = -0.55$, $A_2 = 1.7$, $B_2 = 4.025$, $m = 2.1$, $n = 4.575$, $\rho = 2.03672$, $\alpha = 0.885312$, $\beta = 2.904804$, and the equations (A), or (A'), will be found to be satisfied.

Second General Problem.

" To draw a Sphere tangent to four given Spheres."

Let the radii of the given spheres be denoted by r_1, r_2, r_3, r_4, and the co-ordinates of their centres by (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), (a_4, b_4, c_4); let ρ $(\alpha, \beta, \gamma,)$ be the radius and centre of the required sphere; all being referred to rectangular co-ordinate axes in space.

It is evident that by an analysis similar to that for the circles, the distance of the centre $(\alpha, \beta, \gamma,)$ from each of the centres (a_1, b_1, c_1), &c., would be found equal to the sum or difference of the corresponding radii, according as the spheres had external or internal contact; hence to determine the four unknown quantities, $\alpha, \beta, \gamma, \rho$, we should have the four equations

$$(\text{C.})\quad \begin{aligned}(\alpha - a_1)^2 + (\beta - b_1)^2 + (\gamma - c_1)^2 &= (\varrho \pm r_1)^2,\\ (\alpha - a_2)^2 + (\beta - b_2)^2 + (\gamma - c_2)^2 &= (\varrho \pm r_2)^2,\\ (\alpha - a_3)^2 + (\beta - b_3)^2 + (\gamma - c_3)^2 &= (\varrho \pm r_3)^2,\\ (\alpha - a_4)^2 + (\beta - b_4)^2 + (\gamma - c_4)^2 &= (\varrho \pm r_4)^2.\end{aligned}$$

Or if, to embrace all the cases in one solution, we put $R_1 = \pm r_1$, $R_2 = \pm r_2$, &c., the preceding equations may be written

$$(\text{C}'.)\quad \begin{aligned}(\alpha - a_1)^2 + (\beta - b_1)^2 + (\gamma - c_1)^2 &= (\varrho - R_1)^2,\\ (\alpha - a_2)^2 + (\beta - b_2)^2 + (\gamma - c_2)^2 &= (\varrho - R_2)^2,\\ (\alpha - a_3)^2 + (\beta - b_3)^2 + (\gamma - c_3)^2 &= (\varrho - R_3)^2,\\ (\alpha - a_4)^2 + (\beta - b_4)^2 + (\gamma - c_4)^2 &= (\varrho - R_4)^2.\end{aligned}$$

Let

$$(1.)\quad \alpha = A_1\,\varrho + A_2, \qquad \beta = B_1\,\varrho + B_2, \qquad \gamma = C_1\,\varrho + C_2.$$

Then, by an analysis precisely similar to that employed in the general problem of the contact of circles, the following equations will be found for determining the values of A_1, B_1, C_1, A_2, B_2, C_2.

For the sake of abbreviation, let

$$(2.)\quad \begin{aligned}M_2 &= a_1^2 - a_2^2 + b_1^2 - b_2^2 + c_1^2 - c_2^2 - (R_1^2 - R_2^2),\\ M_3 &= a_1^2 - a_3^2 + b_1^2 - b_3^2 + c_1^2 - c_3^2 - (R_1^2 - R_3^2),\\ M_4 &= a_1^2 - a_4^2 + b_1^2 - b_4^2 + c_1^2 - c_4^2 - (R_1^2 - R_4^2);\end{aligned}$$

then

$$(3.)\quad \begin{aligned}(a_1 - a_2)\,A_1 + (b_1 - b_2)\,B_1 + (c_1 - c_2)\,C_1 &= R_1 - R_2,\\ (a_1 - a_3)\,A_1 + (b_1 - b_3)\,B_1 + (c_1 - c_3)\,C_1 &= R_1 - R_3,\\ (a_1 - a_4)\,A_1 + (b_1 - b_4)\,B_1 + (c_1 - c_4)\,C_1 &= R_1 - R_4.\end{aligned}$$

$$(4.)\quad \begin{aligned}(a_1 - a_2)\,A_2 + (b_1 - b_2)\,B_2 + (c_1 - c_2)\,C_2 &= \tfrac{1}{2}\,M_2,\\ (a_1 - a_3)\,A_2 + (b_1 - b_3)\,B_2 + (c_1 - c_3)\,C_2 &= \tfrac{1}{2}\,M_3,\\ (a_1 - a_4)\,A_2 + (b_1 - b_4)\,B_2 + (c_1 - c_4)\,C_2 &= \tfrac{1}{2}\,M_4.\end{aligned}$$

In order to abridge the expressions for A_1, A_2, &c., resulting from these equations, let

$$(5.)\quad P_a = \frac{(b_1 - b_3)(c_1 - c_4) - (b_1 - b_4)(c_1 - c_3)}{(b_1 - b_2)(c_1 - c_3) - (b_1 - b_3)(c_1 - c_2)},$$
$$Q_a = \frac{(b_1 - b_4)(c_1 - c_2) - (b_1 - b_2)(c_1 - c_4)}{(b_1 - b_2)(c_1 - c_3) - (b_1 - b_3)(c_1 - c_2)}.$$

Changing b into a, and a into b, in these expressions we have

$$(6.)\quad P_b = \frac{(a_1 - a_3)(c_1 - c_4) - (a_1 - a_4)(c_1 - c_3)}{(a_1 - a_2)(c_1 - c_3) - (a_1 - a_3)(c_1 - c_2)},$$
$$Q_b = \frac{(a_1 - a_4)(c_1 - c_2) - (a_1 - a_2)(c_1 - c_4)}{(a_1 - a_2)(c_1 - c_3) - (a_1 - a_3)(c_1 - c_2)}.$$

Changing again c into b, and b into c, in the last formulæ, we have

$$(7.)\quad P_c = \frac{(a_1 - a_3)(b_1 - b_4) - (a_1 - a_4)(b_1 - b_3)}{(a_1 - a_2)(b_1 - b_3) - (a_1 - a_3)(b_1 - b_2)},$$
$$Q_c = \frac{(a_1 - a_4)(b_1 - b_2) - (a_1 - a_2)(b_1 - b_4)}{(a_1 - a_2)(b_1 - b_3) - (a_1 - a_3)(b_1 - b_2)}.$$

$$(8.)\quad \therefore\ A_1 = \frac{(R_1 - R_2)P_a + (R_1 - R_3)Q_a + R_1 - R_4}{(a_1 - a_2)P_a + (a_1 - a_3)Q_a + a_1 - a_4},$$
$$B_1 = \frac{(R_1 - R_2)P_b + (R_1 - R_3)Q_b + R_1 - R_4}{(b_1 - b_2)P_b + (b_1 - b_3)Q_b + b_1 - b_4},$$
$$C_1 = \frac{(R_1 - R_2)P_c + (R_1 - R_3)Q_c + R_1 - R_4}{(c_1 - c_2)P_c + (c_1 - c_3)Q_c + c_1 - c_4}.$$

$$(9.)\quad A_2 = \tfrac{1}{2}\,\frac{M_2 P_a + M_3 Q_a + M_4}{(a - a_2)P_a + (a_1 - a_3)Q_a + a_1 - a_4},$$
$$B_2 = \tfrac{1}{2}\,\frac{M_2 P_b + M_3 Q_b + M_4}{(b_1 - b_2)P_b + (b_1 - b_3)Q_b + b_1 - b_4},$$
$$C_2 = \tfrac{1}{2}\,\frac{M_2 P_c + M_3 Q_c + M_4}{(c_1 - c_2)P_c + (c_1 - c_3)Q_c + c_1 - c_4}.$$

Equations (1) may be written as follows:

$$(10.)\quad \begin{aligned} \alpha - a_1 &= A_1(\varrho - R_1) + k, \\ \beta - b_1 &= B_1(\varrho - R_1) + m, \\ \gamma - c_1 &= C_1(\varrho - R_1) + n, \end{aligned}$$

by making

$$k = A_1\, R_1 + A_2 - a_1,$$
(11.) $$m = B_1\, R_1 + B_2 - b_1,$$
$$n = C_1\, R_1 + C_2 - c_1.$$

Then the first of equations (C′) becomes

$$(\varrho - R_1)^2 - \frac{2\,(k\,A_1 + m\,B_1 + n\,C_1)}{1 - (A_1^2 + B_1^2 + C_1^2)}\,(\varrho - R_1) = \frac{k^2 + m^2 + n^2}{1 - (A_1^2 + B_1^2 + C_1^2)}.$$

Hence, by putting

$$G = (m\,A_1 - k\,B_1)^2 + (n\,A_1 - k\,C_1)^2 + (n\,B_1 - m\,C_1)^2,$$

we get

(12.) $$\varrho = R_1 + \frac{k\,A_1 + m\,B_1 + n\,C_1 \pm \sqrt{k^2 + m^2 + n^2 - G}}{1 - (A_1^2 + B_1^2 + C_1^2)}.$$

Thus the problem is solved analytically by means of equations not transcending the *second degree.*

The number of cases of a sphere touching four given spheres may be determined as follows:

The required sphere may touch the four given spheres all externally, 1 case; or all internally, 1 case; or two internally, two externally, 6 cases; or three externally, one internally, 4 cases; or three internally, one externally, 4 cases; making sixteen cases, as found by Major Alvord, all included in equations (C′).

I must not omit to notice what would otherwise seem to be a defect in the preceding solutions. In both problems there seem to be exceptional cases to which the formulæ are inapplicable. For example, in the tangencies of circles, if $b_1 = b_2 = b_3$, then it appears that $A_1 = \frac{0}{0}$, $A_2 = \frac{0}{0}$, and it is impossible to determine ϱ, α, β, by the proposed method. The reason of this is obvious; for the assumed equations (10), in the first problem, were only employed because the *two equations* (11) contained *three* unknown quantities, α, β, ϱ, and were, therefore, *indeterminate.* But when $b_1 = b_2 = b_3$,

one of the unknown quantities, β, disappears from the equations (11); hence the assumed equations (10) are inapplicable, and, at the same time, they are not wanted; for α and ϱ are then determined by the two equations (11), each of the *first degree*, and then β is found from either of the equations (A′) of the *second degree*. The same considerations apply to the corresponding case of the second general problem, when, for example, $c_1 = c_2 = c_3 = c_4$, in which case $P_a = \frac{0}{0}$, $Q_a = \frac{0}{0}$, $P_b = \frac{0}{0}$, $Q_b = \frac{0}{0}$; hence A_1, B_1, A_2, and B_2 are indeterminate. But under these conditions the unknown quantity γ disappears from the three equations of the *first degree* obtained in the early part of the process; hence the remaining quantities, α, β, ϱ, are easily found, and thence γ from one of the equations (C′), of the *second degree*. Thus in every case the problems are brought within the resolution of equations not surpassing the second degree. I might now give various applications of the second problem which I have computed, but omit them for fear of occupying too much space in the Monthly unnecessarily.

INSTANCES OF NEARLY COMMENSURABLE PERIODS IN THE SOLAR SYSTEM.

By Daniel Kirkwood, Professor of Mathematics, Indiana University.

The following instances of nearly commensurable periods in the solar system have not, I believe, been previously noticed: —

For the 3d and 4th satellites of Jupiter we have

$$7\,P^{\mathrm{III.}} - 3\,P^{\mathrm{IV.}} = 22\,m\;24\,s = \tfrac{1}{1071}\,P^{\mathrm{IV.}}; \ldots\ldots (a).$$

For the Uranian System,

$$13\,P^{\mathrm{II.}} - 4\,P^{\mathrm{IV.}} = 26\,m\;43\,s = \tfrac{1}{726}\,P^{\mathrm{IV.}}; \ldots\ldots (b).$$

$$10\,P^{\mathrm{III.}} - 21\,P^{\mathrm{II.}} = 51\,m\;6\,s = \tfrac{1}{245}\,P^{\mathrm{III.}}; \ldots\ldots (c).$$

For the asteroids and major planets,

11 periods of Mars	=	5	periods of	Astræa.
1 period of Jupiter	=	3	"	Calliope.
3 periods "	=	10	"	Clio.
4 " "	=	11	"	Virginia.
6 " "	=	13	"	Doris.
8 " "	=	29	"	Ariadne.
9 " "	=	22	"	Polyhymnia.
1 period of Saturn	=	9	"	Ariadne.
2 periods "	=	13	"	Leda.

We have also,

8 periods of Jupiter	=	7	periods of	TUTTLE's Comet.
6 " Saturn	=	13	"	"
31 " "	=	12	"	HALLEY's Comet.
6 " Neptune	=	13	"	"

Remarks. — The periods used in (*a*) are taken from HERSCHEL'S Outlines; those in (*b*) and (*c*) from the Proceedings of the American Association for the Advancement of Science, 1854, p. 55. The periods of the asteroids are taken from the Mathematical Monthly for February, 1859. The period of TUTTLE'S Comet is adopted from GOULD'S Astronomical Journal, No. 118. That of HALLEY'S is the mean of the six periods from 1378 to 1835. The epochs at which Neptune must have great perturbing influence on the motion of HALLEY'S Comet are separated by intervals of about 988 years. The last occurred in the early part of the fifteenth century; — the next will be about the beginning of the twenty-fifth. It may also be remarked that the planets Jupiter and Uranus present a case of almost exact commensurability. *Eighty-five times the period of the former, minus twelve times that of the latter*, forms a quantity which amounts to only $\frac{1}{1102}$ of the period of Uranus. Hence in about $1008\frac{1}{4}$ years from the epoch of any conjunction of these planets, another conjunction will occur only *twenty-three minutes* from the place of the former.

THE ELEMENTS OF QUATERNIONS.

[Continued from Page 101.]

IV. — Scalars and Vectors.

19. If $\angle q = \frac{\beta}{\alpha} = 0°$, or $= 180°$, then β has either the same direction as α or the opposite, and q degenerates into either a positive or negative real number, and is then called a *Scalar*. Like tensors, scalars may be applied to any line in space without regard to its direction; and like them, they are commutative in combination with any quaternion. The product of scalars is evidently a scalar; and the conjugate of a scalar does not differ from the scalar itself.

20. If $\angle q = \pm 90°$, then q is called a *Vector*. Vectors will be denoted by v, v', &c. It is evident from (4) and (3′) that

(16) $$\mathrm{K}v = -v.$$

If $\mathrm{T}v = m$, then by (11),

(17) $$v^2 = -m^2;$$

that is, *the square of a vector is a negative scalar.* The square root of (17), $v = m\sqrt{-1}$, may be regarded as the expression for a vector of *indeterminate axis*, whose tensor is m. When $m = 1$, $v = \sqrt{-1}$ is called a *unit vector*. $\sqrt{-1_q}$ may be used to denote this vector, when its axis is fixed in the direction of $\mathrm{Ax}.q$. The geometrical signification then of $\sqrt{-1}$ in this system, is the operation of turning any line to which it may be applied through an angle of 90° around any axis perpendicular to that line. Any quaternion may be expressed as a power of a vector, since, by (11), if, m being a positive number,

(18) $$q = [m\sqrt{-1_q}]^n, \qquad \text{then } \mathrm{T}q = m^n, \qquad \angle q = n\,90°,$$

and the independent real quantities m and n are sufficient to determine $\mathrm{T}q$ and $\angle q$.

21. The lines α and β being given, let the directions of the lines γ and δ be respectively parallel and perpendicular to that of α, and

let their lengths be such that $\gamma + \delta = \beta$. These conditions completely determine γ and δ, both of which will be in the plane of α and β. Then, by § 8,

$$q = \beta \div \alpha = (\gamma + \delta) \div \alpha = \gamma \div \alpha + \delta \div \alpha;$$

but $\gamma \div \alpha$ is a scalar, and $\delta \div \alpha$ is a vector. Any given quaternion then may be resolved into a sum of two other determinate quaternions, one of which is a scalar, and the other a vector; in this case the former is called the *scalar of the given quaternion*, and the latter *its vector*. The scalar of a quaternion, q, is written $\mathrm{S}q$, and its vector $\mathrm{V}q$; thus

(19) $$q = \mathrm{S}q + \mathrm{V}q = \mathrm{V}q + \mathrm{S}q.$$

From § 19, (16), and (4) we obtain

(20) $$\mathrm{KS}q = \mathrm{S}q = \mathrm{SK}q, \quad \mathrm{KV}q = -\mathrm{V}q = \mathrm{VK}q;$$ whence

(19 *a*) $$\mathrm{K}q = \mathrm{S}q - \mathrm{V}q.$$

The sum and difference of (19) and (19 *a*) gives

(21) $$q + \mathrm{K}q = 2\,\mathrm{S}q, \quad \text{and } q - \mathrm{K}q = 2\,\mathrm{V}q;$$ or

symbolically, $\mathrm{S} = \frac{1}{2}(1 + \mathrm{K})$, and $\mathrm{V} = \frac{1}{2}(1 - \mathrm{K})$.

If two or more quaternions are equal, their scalars must be equal, and also their vectors; and conversely if their scalars and vectors are equal each to each, the quaternions must be equal.

22. Let the planes of the two vectors, v and v', intersect in the line α; and let $v\,\alpha = \beta$, and $v'\,\alpha = \beta'$; then both β and β' will be in the plane of $\mathrm{Ax.}\,v$ and $\mathrm{Ax.}\,v'$, and, if $\gamma = \beta + \beta'$, $(v + v') = \gamma \div \alpha$. But $\gamma \div \alpha$ *is a vector*, whose axis, in the plane of $\mathrm{Ax.}\,v$ and $\mathrm{Ax.}\,v'$, makes the same angles with these axes that the line γ does with the lines β and β' respectively, and whose tensor bears the same ratios to $\mathrm{T}v$ and $\mathrm{T}v'$ that $\mathrm{T}\gamma$ does to $\mathrm{T}\beta$ and $\mathrm{T}\beta'$ respectively. If then we set off on $\mathrm{Ax.}\,v$ and $\mathrm{Ax.}\,v'$ lines whose tensors are equal

respectively to $\mathrm{T}v$ and $\mathrm{T}v'$, a parallelogram formed on these two sides will be similar to the parallelogram formed on β and β', and its diagonal corresponding to the diagonal γ of the latter parallelogram, will be the axis of $(v + v')$, and the tensor of this diagonal will be equal to the tensor of $(v + v')$. Hereafter we shall consider *the axis of a vector as a line of definite length, whose tensor is equal to the tensor of the vector.* With this definition, then,

(22) $$\Sigma \mathrm{Ax}.\, v = \mathrm{Ax}.\, \Sigma v.$$

23. Let the line β be the intersection of the planes of v' and v, and such that rotation around it is positive from $\mathrm{Ax}.v'$ to $\mathrm{Ax}.v$; and let $v' = \alpha \div \beta$, and $v = \beta \div \gamma$; then both α and γ will be in the plane of $\mathrm{Ax}.v'$ and $\mathrm{Ax}.v$, and if we put

(23) $$v'v = \alpha \div \gamma = q; \text{ then } \mathrm{Ax}.q = \beta, \text{ and } \angle q = 180^\circ - \frac{\mathrm{Ax}.v}{\mathrm{Ax}.v'}.$$

Since $\mathrm{U}v'v^{-1} = \mathrm{U}(\alpha \div \beta)\,\mathrm{U}(\beta \div -\gamma) = \mathrm{U}(\alpha \div -\gamma)$, if we put

(23 a) $$v'v^{-1} = -\mathrm{T}v^{-2}.\,q = q'; \text{ then}$$
$$\mathrm{Ax}.q' = \beta, \quad \text{and } \angle q' = 180^\circ + \angle q = \frac{\mathrm{Ax}.v'}{\mathrm{Ax}.v}.$$

As a special case of (8), we have, by (16),

(24) $$\mathrm{K}v'v = \mathrm{K}v\,.\,\mathrm{K}v' = vv'.$$

24. Let $\mathrm{Ax}.v$ be in the plane of q, and α a line in the direction of $\mathrm{Ax}.q$; and let $v\alpha = \beta$, and $q\beta = \gamma$. Then

(25) $$qv = (\gamma \div \beta)(\beta \div \alpha) = \gamma \div \alpha, \quad \textit{a vector},$$

$\mathrm{T}qv = \mathrm{T}q\,.\,\mathrm{T}v$, and $\mathrm{Ax}.qv$ is so placed in the plane of q that $\frac{\mathrm{Ax}.qv}{\mathrm{Ax}.v} = \frac{\gamma}{\beta} = \angle q$. Therefore,

(26) $$\mathrm{Ax}.qv = q\,\mathrm{Ax}.v.$$

From (25), (16), and (8) we deduce

(27) $$\mathrm{K}qv = -qv = \mathrm{K}v\,.\,\mathrm{K}q = -v\,.\,\mathrm{K}q, \qquad qv = v\,.\,\mathrm{K}q;$$

and putting $\mathrm{K}q$ for q in (27), we get, by (5),

(27 *a*) $$\mathrm{K}q\,.\,v = v\,.\,\mathrm{K}^2 q = v q\,;$$

whence, making the same substitution in (26), we get by (27 *a*)

(26 *a*) $$\mathrm{Ax.}\,v q = \mathrm{K}q\,.\,\mathrm{Ax.}\,v.$$

Putting $q v = v'$, (26) becomes $\mathrm{Ax.}\,v' = q\,\mathrm{Ax.}\,v$; whence

(28) $q = v' \div v = v' v^{-1} = \mathrm{Ax.}\,v' \div \mathrm{Ax.}\,v$, or, putting v^{-1} for v,

(28 *a*) $$v' v = \mathrm{Ax.}\,v' \div \mathrm{Ax.}\,v^{-1}.$$

25. Let the line α^{-1}, the *reciprocal* of the line α, be defined by the equations

(29) $\alpha = \mathrm{Ax.}\,v,\quad \alpha^{-1} = \mathrm{Ax.}\,v^{-1}$; whence $(\mathrm{Ax.}\,v)^{-1} = \mathrm{Ax.}\,v^{-1}$.

Equations (26) and (28 *a*) give respectively by the substitution of (29)

(30) $$\mathrm{Ax.}\,q v^{-1} = q\,(\mathrm{Ax.}\,v)^{-1},$$

(31) $$v' v = \mathrm{Ax.}\,v' \div (\mathrm{Ax.}\,v)^{-1} = \mathrm{Ax.}\,v'\,.\,\mathrm{Ax.}\,v.$$

26. The results expressed in (22), (28), (31), (26), and (30) show that a vector, v, is, under the definitions assumed, (in § 22 and (29),) perfectly represented in these cases by a line, $\mathrm{Ax.}\,v$. Thus in § 22 the sum of vectors is proved *to be a vector*, and (22) expresses that the axis of this vector may be found by adding the axes of the given vectors. Equation (28) shows that the quotient of two vectors is the same quaternion as the quotient of their axes. Equation (31) shows that the definition (29) is so taken as to represent the multiplication of two vectors by the multiplication of their axes. Lastly, (25) shows that the product $q v$ is, in that special case, *a vector*, and (26) shows that the axis of this vector may be found by multiplying $\mathrm{Ax.}\,v$ by q; while (30) shows that the same property is true of $q v^{-1}$. The only possible cases remaining are the forms $q + v$, $q v$ and $q v^{-1}$, (except the special cases (25) and (30),) $v q$ and $v^{-1} q$;

and since these have, as yet, no meaning when a line, as Ax. v, is put in the place of v, we may now give them an arbitrary meaning in accordance with the conclusion, deduced from the other cases, that Ax. v may *always* be substituted for v; that is, that *a line is a vector*, and that *whatever may be proved of lines is proved of the vectors of which they are the axes.* HAMILTON indeed introduces the term *vector* as a name for a straight line, deriving it from *vehere*, because a line is supposed *to carry* a point from one of its extremities to the other. Afterwards, assuming its other signification, as a special case of a quaternion, he proceeds somewhat in the reverse order to that in which the subject is developed here.

27. Any vector, v, may, by § 22, be expressed as the sum of three mutually rectangular vectors, v_x, v_y, and v_z, whose axes are equal in length to the projections* of Ax. v in their respective directions. If the tensors of v_x, &c., are denoted by $(\mathrm{T}\,v)_x$, &c., and their versors by i, j, and k respectively, we have

$$q = \mathrm{V}\,q + \mathrm{S}\,q = (\mathrm{T}\,\mathrm{V}\,q)_x\, i + (\mathrm{T}\,\mathrm{V}\,q)_y\, j + (\mathrm{T}\,\mathrm{V}\,q)_z\, k + (\mathrm{S}\,q),$$

in which the four quantities in parentheses are four independent elements involved in the complete determination of q. *Compare* § 14.

28. The planes of the quaternions $q = \beta \div \alpha$ and $q' = \beta' \div \alpha$ intersect in the line α; and it is evident that $\mathrm{S}\,q\,.\,\alpha$ is the projection of β in the direction of α, and $\mathrm{V}\,q\,.\,\alpha$ its projection on a plane perpendicular to α; whence it follows for the two quaternions, q and q', and thence by easy extension for any number, that

(32) $$\mathrm{S}\,\Sigma = \Sigma\,\mathrm{S}, \qquad \text{and} \qquad \mathrm{V}\,\Sigma = \Sigma\,\mathrm{V}.$$

But, as the value of $\mathrm{S}\,\Pi$ does not depend on what or how many factors the product may be composed of, so long as the product

* For greater completeness we may add that the *projection* of a line in any given direction, that is, on any given line, is so much of the latter line as is intercepted between perpendiculars dropped upon it from the extremities of the former.

itself remains the same; while ΠS evidently does depend on the composition of Π; it follows that we do *not* have in general S $\Pi =$ ΠS. And as § 23 shows that Π V is not necessarily a vector, we do *not* have in general V $\Pi = \Pi$ V.

29. We are now prepared to deduce the following relations, connecting the notation of quaternions with trigonometry. If $q = \beta \div \alpha$,

(33) $\quad \mathrm{S}q\,.\,\mathrm{T}\alpha = \mathrm{T}\beta\,.\cos\frac{\beta}{\alpha}, \quad \mathrm{S}q = (\mathrm{T}\beta \div \mathrm{T}\alpha)\cos\frac{\beta}{\alpha} = \mathrm{T}q\,.\cos\frac{\beta}{\alpha},$

(33a) $\mathrm{V}q.\mathrm{T}\alpha = \mathrm{TV}q.\mathrm{T}\alpha.\mathrm{UV}q = \mathrm{T}\beta.\sin\frac{\beta}{\alpha}.\sqrt{-1_q}, \quad \mathrm{V}q = \mathrm{T}q.\sin\frac{\beta}{\alpha}.\sqrt{-1_q};$

whence

(34) $\quad q = \mathrm{T}q(\cos\frac{\beta}{\alpha} + \sin\frac{\beta}{\alpha}.\sqrt{-1_q}), \qquad \mathrm{U}q = \cos\frac{\beta}{\alpha} + \sin\frac{\beta}{\alpha}.\sqrt{-1_q},$

and

(35) $\quad (\mathrm{S}q)^2 - (\mathrm{V}q)^2 = \mathrm{T}q^2\left(\cos^2\frac{\beta}{\alpha} + \sin^2\frac{\beta}{\alpha}\right) = \mathrm{T}q^2 = (\mathrm{S}q)^2 + (\mathrm{TV}q)^2.$

The second of equations (34) is equivalent to the two equations

(34′) $\quad \mathrm{SU}q = \cos\frac{\beta}{\alpha}, \qquad \mathrm{VU}q = \sin\frac{\beta}{\alpha}\,.\sqrt{-1_q}.$

The latter of these is equivalent to the two equations

(34″) $\quad \mathrm{TVU}q = \pm\sin\frac{\beta}{\alpha},$* $\quad \mathrm{UVU}q = \pm\sqrt{-1_q}.$

By the substitution of (34′) in (33) and (33 a), we obtain

(36) $\quad \mathrm{S}q = \mathrm{T}q\,.\,\mathrm{SU}q, \qquad \mathrm{V}q = \mathrm{T}q\,.\,\mathrm{VU}q.$

Finally, by means of (34), (11), and (34′), we get

(37) $\quad \mathrm{U}q^m = (\cos\frac{\beta}{\alpha} + \sin\frac{\beta}{\alpha}\,.\sqrt{-1_q})^m = \cos m\frac{\beta}{\alpha} + \sin m\frac{\beta}{\alpha}\,.\sqrt{-1_q},$

which coincides with the expression of De Moivre's *Theorem.*

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* Tensors being always positive, if $\frac{\beta}{\alpha} > 180$, it would be necessary to use the lower signs.

[To be Continued.]

NOTES ON THE THEORY OF PROBABILITIES.

By Simon Newcomb, Cambridge, Mass.

[Continued from Page 335, Vol. I.]

15. The problem which we are about to consider is one of the most fruitful in the theory of probabilities, as out of it grow the theory of errors, the theory of chance distribution, the law of averages, and the estimation of the probability that an observed concurrence of events is the result of a law of nature.

To find the probability that an event of which the probability on a single trial is p *will happen* s *times on* n *trials.*

The probability that it will fail on every trial is $(1-p)^n$, $1-p$ being the probability that it will fail on any single trial.

The probability that it will happen on the first trial and fail on the $n-1$ following ones is $p(1-p)^{n-1}$. But as the single event is as likely to occur on the 2d, 3d, ... nth trial as on the first, the probability that it will occur just once is $np(1-p)^{n-1}$.

The probability that the event will occur on the first two trials and fail on the $n-2$ subsequent ones is $p^2(1-p)^{n-2}$. But the two events can equally occur on the (1, 3), (1, 4) . . . . . (1, n), or the (2, 3), (2, 4), &c. trials; in fact there will be $\overset{2}{\underset{n}{C}}$ pairs of trials on which the two events can occur, so that $\overset{2}{\underset{n}{C}} p^2 (1-p)^{n-2}$ is the probability that it will occur twice.

By a process of reasoning exactly like the last, we find the probability that it will occur s times to be

$$P_s = \overset{s}{\underset{n}{C}} p^s (1-p)^{n-s}, \tag{1}$$

which is the $(s+1)$st term in the development of the binomial $[(1-p)+p]^n$. The sum of the probabilities of all the possible results of the n trials is therefore 1, as it ought to be.

As an example to elucidate the above, suppose that a cent is so formed that a head is twice as likely to be thrown as a tail, so that the probability of the former on each throw is $\frac{2}{3}$. If the coin is thrown four times, the results of the four throws may be as follows. After each separate result is written the fraction expressing the probability of that result.

No heads, $tttt\ \frac{1}{3^4} = \frac{1}{81}$,
1 head, $httt$, $thtt$, $ttht$, $ttth$, each $\frac{2}{81}$; $\times\, 4 = \frac{8}{81}$,
2 heads, $hhtt$, $htht$, $htth$, $thht$, $thth$, $tthh$, $6 \times \frac{4}{81} = \frac{24}{81}$,
3 heads, $hhht$, $hhth$, $hthh$, $thhh$, $= \frac{32}{81}$,
4 heads, $hhhh = \frac{16}{81}$.

If we supposed heads as likely to be thrown as tails, we should find these probabilities to be $\frac{1}{16}$, $\frac{4}{16}$, $\frac{6}{16}$, $\frac{4}{16}$, $\frac{1}{16}$, respectively. The result would evidently be the same if we supposed that four coins were thrown from a box together.

16. To resume the general discussion, let us see what value of s is the most probable. This value we will determine by the condition that its probability must be greater than that of the next smaller number, and also greater than that of the next greater number, or

$$\overset{s}{\underset{n}{C}} p^s (1-p)^{n-s} > \overset{s-1}{\underset{n}{C}} p^{s-1} (1-p)^{n-s+1};$$
$$\overset{s}{\underset{n}{C}} p^s (1-p)^{n-s} > \overset{s+1}{\underset{n}{C}} p^{s+1} (1-p)^{n-s-1}.$$

Since $\quad \overset{s}{\underset{n}{C}} = \frac{n-s+1}{s} \overset{s-1}{\underset{n}{C}}; \qquad \overset{s+1}{\underset{n}{C}} = \frac{n-s}{s+1} \overset{s}{\underset{n}{C}};$

we have from the division of the first inequality by $\overset{s-1}{\underset{n}{C}} p^{s-1} (1-p)^{n-s}$,

$$1 - p < \frac{n-s+1}{s} p, \qquad \text{which gives } s < p(n+1);$$

and from the division of the second inequality by $\overset{s}{\underset{n}{C}} p^s (1-p)^{n-s-1}$ we have

$$1 - p > \frac{n-s}{s+1} p, \qquad \text{which gives } s > p(n+1) - 1;$$

s is therefore the greatest whole number in $p(n+1)$. If s and n are very large numbers, we have very nearly

$$(2) \qquad \frac{s}{n} = p.$$

It follows, therefore, that in a great number of trials events are more likely to occur a number of times proportional to their respective probabilities than any other number. Thus if a cent is thrown one hundred times, heads are more likely to be thrown fifty than any other single number of times. But it must not be supposed that they are therefore more likely to be thrown fifty times than not, for it is *almost* as likely to be thrown 49 times or 51 times as fifty times. The chances that it would be thrown *exactly* fifty times would be quite small, because there are so many other numbers that might be thrown.

17. Another deduction from the expression (1) is the following: however small the probability of an event on a single trial, by increasing the number of trials we can render the probability that the event will occur at least once as great as we please. For the probability that it will fail on every one of the n trials being $(1-p)^n$; however small p may be, we may make n so great that $(1-p)^n$ shall be as small as we please.

18. Suppose now that n is infinitely great, and p infinitely small, and that $np = \alpha$, α being a finite quantity. We may then put $n = n - 1 = n - 2$, &c. We shall then have, while s is finite,

$$\overset{s}{\underset{n}{C}} p^s = \frac{n^s p^s}{s!} = \frac{\alpha^s}{s!},$$

$$(1-p)^{n-s} = (1-p)^n = (1-p)^{\frac{\alpha}{p}} = e^{-\alpha},$$

e being the Napierian base. Substituting these values in (1) we obtain for the probability that the event will occur s times

(3) $$P_s = \frac{\alpha^s e^{-\alpha}}{s!}.$$

The probability that the event will fail, is therefore $e^{-\alpha}$; that it will occur once only, $\alpha e^{-\alpha}$; twice only, $\frac{\alpha^2}{2} e^{-\alpha}$, &c.

The sum of this series of probabilities continued to infinity is

$$e^{-\alpha}\left(1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \&c.,\right) = e^{-\alpha} \,.\, e^{\alpha} = 1, \text{ as it ought to be.}$$

19. We may apply this equation to the determination of the probability that, if the stars were scattered at random over the heavens, any small space selected at random would contain s stars. Let N be the whole number of stars, h the number of units of space in the heavens, then $\frac{N}{h}\, dh$ may be taken to represent the infinitely small probability that the infinitely small space dh contains a star. Moreover, if l represents the extent of space selected at random which we consider, we may consider the examination of each dh as a trial, and the number n of trials will then be $\frac{l}{dh}$. The value of α will then become $\frac{N}{h} l$, and by substitution in (3) we have

(4) $$P = \frac{N^s l^s e^{-\frac{Nl}{h}}}{h^s s!}$$

for the probability that the space l contains s stars. Suppose, as a numerical example, that l is a square degree, and $N = 1500$, which is about the number of stars of the fifth and higher magnitudes; $s = 6$. We then have $\frac{l}{h} = \frac{1}{41253}$; by the substitution of these values in (4) we shall have the probability that any square degree selected at random in the heavens contains six stars. Multiplying this probability by 41253, the number of square degrees in the whole heavens, we obtain the probability that, *if the heavens were divided at random into square degrees, some one of those square degrees would contain six stars.* This probability we find to be

$$\frac{1500^6}{41253^5 \, . \, 6\,!} \cdot e^{-.03636} = .000000128.$$

This, however, is evidently rather smaller than the probability that six stars should be found so near together that a square degree could be fitted on so as to include them.

Mr. Mitchell committed an error in his solution, the effect of which, if I mistake not, is to make his probability too great. His general method is, however, better applicable to this particular problem than that given above, but as there is a margin of vagueness and uncertainty about the problem in question, so that the answer does not admit of being expressed in exact numbers without an excessively complicated process of reasoning, I have preferred to deduce an approximate solution from the general formulæ, to be used in so many more problems.

20. Let us now consider Prof. Forbes's objections to the above results of the calculus of probabilities.

He scattered paint from a brush upon a wall, and found double and triple spots and groups innumerable. This is about as decisive as an attempt to disprove the Pythagorean proposition by measuring the squares described on a triangle without knowing whether it had or had not a right angle, and finding that one square was not equal to the sum of the others. As a mathematician would answer this objection by saying that his result only proved that he had either made a mistake in his measurements, or had not measured a right triangle; so Prof. Forbes's result only proves that either he was mistaken as to the marked character of the groupings, or that the proximity of the components of each group was the effect of their positions being determined *by the action of the same cause,* which is all that the theory of probabilities claims for the Pleiades. The latter supposition is by no means improbable, because a group of spots might be formed by the breaking up of a drop after it had left the brush.

21. Prof. FORBES remarked that an exactly uniform distribution of stars would not be expected as the result of a random distribution. In this he is correct; and as it is an interesting problem, we shall here determine what law a random distribution may be expected to follow. It may appear paradoxical to assert that the results of chance can be expected to follow any law; but such is really the case, and the formula (3) determines the law. As an example, suppose that the heavens are divided into 1500 equal portions, and that 1500 stars are distributed at random, or, to speak with more philosophical accuracy, that the causes which determine the position of each separate star are entirely independent of those which determine the position of any other. Then by reasoning as in § 19 we find $\alpha = 1$; and by formula (3) the probability that a unit of space selected at will contains no star, will be $\frac{1}{e} = \frac{1}{2.718\ldots}$*; one star, $\frac{1}{e}$; two stars, $\frac{1}{2e}$; three stars, $\frac{1}{2.3e}$; &c. If we then select the whole 1500 units we ought to expect the number which would be found to contain the several numbers of stars to be somewhere near 1500 multiplied by the respective probabilities, or

about	$\frac{1500}{e} = 552$	portions	containing	no star,
"	$\frac{1500}{e} = 552$	"	"	1 star,
"	$\frac{1500}{2e} = 276$	"	"	2 stars,
"	$\frac{1500}{2.3e} = 92$	"	"	3 stars,
"	$\frac{1500}{4!e} = 23$	"	"	4 stars,
"	$\frac{1500}{5!e} = 4(+)$	"	"	5 stars,
"	$\frac{1500}{6!e} = 1$	"	"	6 stars,

* The acute reader will perceive that the solutions in §§ 19 and 21 are those of a problem slightly (though not materially) different from that actually propounded.

and it would be quite improbable (about 1 chance to 8) that any space would be found to contain more than six stars.

If any one wishes an experimental illustration of the principle let him take a pint of rice, color a hundred grains of it black, mix the black grains thoroughly with the remainder, and stir the mixture till he finds six or eight of the black grains to form a group by themselves. Before this result arrives he will in all probability be willing to admit that should he ever see such a group in such a mixture, he would not believe that it was formed by indiscriminate mixing.

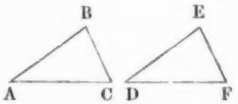
A SECOND BOOK IN GEOMETRY.

[Continued from Page 104.]

CHAPTER VI.

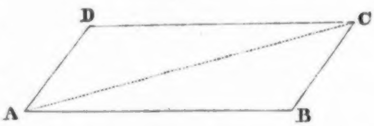
THE PYTHAGOREAN PROPOSITION.

91. *Theorem.* If a triangle has one side and the adjacent angles equal respectively to a side and the adjacent angles in another triangle, the two triangles are equal. — *Proof.* Let us suppose that, in the triangles A B C and D E F, we have the side A B equal to the side D E, the angle at A equal to the angle D, and that at B equal to that at E. Let us imagine the triangle D E F to be laid upon A B C in such a manner as to place E upon B, and D upon A, which can be done because A B is equal to D E. Now, as the angle A is equal to D, the line D F will run in the same direction as A C, and, as it starts from the same point, will coincide with it. Also, since the angle B is equal to E, the line E F will coincide with B C. Whence, by article 90, the triangles are equal.



92. *Theorem.* The opposite sides of a parallelogram are equal. — *Proof.* Article 90 gives us the only test of geometrical equality. So that in order to prove this theorem we must show that in a parallelogram like A B C D, A B may be made to coincide with D C, and B C with A D. And this would evidently be done if we could show that the triangle A B C is equal to A D C. But in these triangles the line A C is the same, and by article 87 the adjacent angles A C B and C A B are equal to the adjacent angles C A D and A C D; whence, by article 91, the two triangles are equal, and A D is equal to B C, and A B equal to D C.

Fig. B



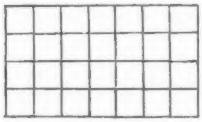
93. *Axiom.* If one end of a straight line stands still while the other turns round, the end that moves will *begin* to move in a direction at right angles to that of the line itself. Thus if A B were to begin to turn about the point A, B would *begin* to move either towards C or towards D. [If this proposition is not acknowledged as an axiom, the proof is in Select Propositions, Nos. 12 – 16.]

94. *Theorem.* The angles of a triangle cannot be altered without altering the length of the sides. — *Proof.* If in any triangle, as A B C, the sides were unchangeable, any alteration of the angles A and B would by article 93 make the point C move in two directions at once, (namely, at right angles to A C, and at right angles to B C,) which is impossible, and therefore the angles cannot be altered.

95. *Corollary.* If the three sides of a triangle are respectively equal to the three sides of another triangle, the angles of one must be equal to those of the other, and the equal angles are enclosed in the equal sides.

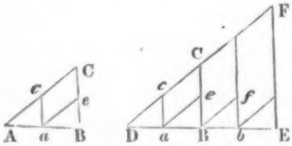
96. *Theorem.* If the opposite sides of a quadrangle are equal, the quadrangle is a parallelogram. — *Proof.* If in the quadrangle A B C D (Fig. B.), the sides A B and C D are equal, and also the sides A D and B C are equal, then, by drawing the diagonal A C, we have the triangles A B C and A D C, composed of equal sides, and, by article 95, the angle D A C must be equal to the angle A C B, and the angle D C A to the angle B A C; whence, by article 89, the figure is a parallelogram.

97. *Theorem.* The area of a rectangle is the product of its length by its breadth. — *Proof.* By drawing lines, at a distance apart equal to the unit of length, parallel to the sides of the rectangle, we shall (arts. 87 – 89) divide the rectangle into little squares, each of which is a unit of surface. Moreover, these squares are arranged in as many rows as there are units of length in one side of the rectangle, each row containing as many squares as there are units of length in the other side; so that the whole number of squares is found by multiplying the length of the rectangle by its breadth.



98. *Scholium.* In the above proof it is taken for granted that the sides of the rectangle can be divided into units of length. This can usually be done by taking the units sufficiently short, that is to say, if the lines are not an even number of inches in length, we may take tenths of an inch as the unit; if they are not even tenths, we can divide them into hundredths, or thousandths, or even millionths, of an inch. If after dividing each line into millionths of an inch anything less than the millionth of an inch were left at either end, it would be too small to be taken into consideration. There would be no error, even in reasoning, from neglecting it. For as long as anything is left at the ends of the lines, we can choose smaller units, but as long as the units are of any size at all, our reasoning holds good, and the rectangle is measured by the product of its dimensions.

99. *Theorem.* If the angles of one triangle are equal to those of another triangle, any two sides of one of the triangles have the same ratio to each other as that of the two sides including the same angle in the other triangle. — *Proof.* Let the triangles A B C and D E F be equiangular with respect to each other. Place the vertex A upon the vertex D, and allow the side A B to fall upon the side D E. Since the angles A and D are equal, the line A C will fall upon the line D F, and since the angles C and F are equal, the line B C will lie parallel to the line E F.



Let the sides A B and D E be divided into units of length, A *a*, *a* B, B *b*, &c. Through the

points of division draw lines $a c$, $b d$, &c., parallel to E F. Draw also the lines $a e$, Bf, &c., parallel to D F. By article 91, the triangles A a c, a B e, &c. are equal. By article 92, $a e$ is equal to c C, Bf to C d, &c. Hence it is easy to see that A B is composed of the same number of times A a, that A C is of A c, and that in like manner D E is as many times D a, as D F is of D c. And thus by article 78, D E : D F = A B : D C, because each of these ratios is equal to B a : a e.

100. *Scholium.* If the lines A B and D E do not consist of a certain number of times the first unit of length which we have chosen, we may choose a unit small enough to make the remainder small enough to be neglected.

101. *Definitions.* The right angle, right triangle, legs, and hypotenuse, are defined in articles 14 and 17.

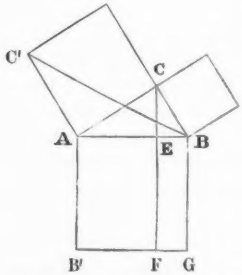
102. *Theorem.* The sum of the three angles of a triangle is equivalent to two right angles. This proposition has been proved in articles 26 – 31, 34 – 36, and 57 – 62.

103. *Corollary.* The sum of the two angles opposite to the legs of a right triangle is equivalent to one right angle.

104. *Corollary.* If an angle opposite a leg in one right triangle is equal to an angle in another right triangle, the two right triangles are equiangular with respect to each other.

105. *Theorem.* If from the vertex of the right angle in a right triangle, a line be drawn at right angles to the hypotenuse dividing the hypotenuse into two segments, each leg is a mean proportional between the whole hypotenuse and the segment nearest the leg. — *Proof.* Let A B C be a right triangle with a right angle at C. Draw C F at right angles to A B. The triangle B E C is right angled at E, and has an angle at B equal that at B in the triangle A B C. Hence, by article 104, the triangle B E C has its angles equal to those of A B C. Hence, by article 99, B E : B C = B C : B A. In the same way A E : A C : : A C : A B.

Fig. A.



106. *Theorem.* The square on the hypotenuse is equivalent to the sum of the squares on the legs. — *Proof.* Let A C B (Fig. A.) be a right triangle, with a right angle at C, and let a square be drawn on each side. Draw C F at right angles to A B. The figure B F will be a rectangle, because all its angles will be right angles. It will, therefore, be measured by the product of B E into E F; or (since E F = B G and B G = B A), by the product of B E × B A. But since B C is a mean proportional between the lines B E and B A, this product is equal to B C × B C, which is the measure of the square on B C. That is, the measure of the rectangle B F is the same as that of the square on B C. In the same manner it may be shown that the rectangle A F is equivalent to the square on A C. But the sum of these two rectangles is evidently equal to the square on the hypotenuse.

107. In these thirty-one articles I have given you a proof of the Pythagorean proposition in the usual synthetic form. Parts of the proof are not completely filled out, but the omitted steps are so short and easy, that I think you will have no difficulty whatever in supplying them. Do not be satisfied with understanding each of the thirty-one articles, but examine them closely from the 76th to the 106th, and see whether I have introduced anything which is not necessary to the proof of 106. In making this examination, it will be most convenient for you to proceed backward.

These thirty-one articles have been here introduced as lemmas, i. e. preparatory propositions, for demonstrating the Pythagorean proposition. But they are also each one truths worth knowing, and will aid in establishing many theorems that have no connection with a right triangle.

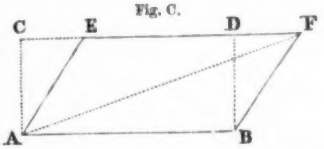
108. Another mode of analyzing this proposition would be suggested by our knowledge of the fact that any triangle is equivalent to half a rectangle of the same base and altitude. I will not lead you through this analysis, but will simply build up for you by synthesis, a

Second Proof of the Pythagorean Proposition.

109. *Definitions.* Any side of a triangle or quadrangle may be called its base, and the altitude of the figure is the distance from the base to the most distant vertex of the figure. This distance is measured by a straight line at right angles to the base, and contained between the vertex and the base, prolonged if need be.

110. *Theorem.* Every parallelogram is equivalent to a rectangle of the same base and altitude. — *Proof.* Let A B C D be a rectangle, and A B E F a parallelogram having the same base A B, and the same altitude B D. It is manifest that if the triangle B D F by which the parallelogram overlaps the rectangle is equal to the triangle A E C by which the rectangle overlaps the parallelogram, the two quadrangles are equivalent. But A E and its adjacent angles are equal to B F and its adjacent angles, and therefore the triangles are equal (Art. 91), and the quadrangles equivalent.

Fig. C.



111. *Theorem.* Every triangle is equivalent to half a rectangle of the same base and altitude. — *Proof.* Let A F B be a triangle, and A B C D a rectangle having the same base, A B and the same altitude B D (Fig. C.). Continue C D to F and draw A E parallel to B F. The triangle A E F has its three sides equal to those of A B F; the triangles are, therefore, equal to each other (Art. 95); and each is equal to half the parallelogram A B E F, which is equivalent to the rectangle A B C D.

112. *Theorem.* The square on the hypotenuse is equivalent to the sum of the squares on the legs. — *Proof.* Having drawn the figure (Fig. A.), as for the former proof, draw the lines C′ B, and B′ C. The triangle A B′ C has the same base A B′, and the same altitude A E as the rectangle A F, and is equivalent to half that rectangle. The triangle A B C′ has the same base A C′ and the same altitude A C as the square C′ C, and is equivalent to half that square. So that if the triangles A B C′ and A B′ C are equal, the rectangle is equivalent to the square. But these triangles are equal, for if A B′ C were turned about the vertex A as on a pivot until the point C covered C′, then B′ would cover B, and the triangles would coincide. For A C would rotate through a right angle, and A B′ through a right angle; and A C = A C′, and A B = A B′.

113. This proof of the Pythagorean proposition is more strictly geometrical than the preceding, as it does not involve the idea of multiplying lines to measure areas. But you must remember that each is equally conclusive. I have here also omitted some of the shorter steps. You should not only be able to fill out these steps when the omission is pointed out to you, but also to discover the omission for yourselves. Take the proofs which I have written down and examine them step by step, asking at each step, is that strictly self-evident? Can it be questioned? Can it be divided into two steps? Is there need of proof? If so, has the proof been given in a previous article? It is only by such an earnest study of the book and of the subject that you can make the process of mathematical reasoning become a sure and pleasant road for you to the discovery of truth.

[To be continued.]

Editorial Items.

THE following students have sent us solutions of the Prize Problems in the October Number, Vol. II., of the Monthly.

MISS AMANDA BENNETT, Red Wing, Minnesota. Prob. I.
MISS HARRIET L. ENSIGN, Catskill Academy, N. Y. Probs. I., II., III., and IV.
MISS HARRIET S. HAZELTINE, Worcester, Mass. Probs. I., II., and IV.
OAKLEY H. SMITH, New Hampton Institution, Fairfax, Vt. Probs. I. and II.
D. Y. BINGHAM, Ellicotteville, N. Y. Prob. III.
SAMUEL S. EASTWOOD, High School, Saxonville, Mass. Probs. I. and II.
C. P. MARSTON, Public High School, Hartford, Ct. Probs. I. and II.
WILLIAM W. JOHNSON, Sophomore Class, Yale College, Ct. Probs. III. and IV., and first part of V.
BENJAMIN F. WEBBER, Wesleyan Seminary, Kent's Hill, Maine. Probs. I. and II.
ISAAC H. TURRELL, Drewersburg, Ind. Probs. I., II., III., and IV.
GEORGE D. HALE, Select School, Adams Centre, N. Y. Probs. I. and II.
F. E. HASTINGS, Literary Institution, Suffield, Ct. Prob. I.
G. W. BROWN and ISAAC FULLER, Collegiate Institution, Bowden, Ga. Each Probs. I., II.
WILLIAM C. CLEVELAND, Lawrence Scientific School, Cambridge, Mass. All but V.
WILLIAM C. HENCK, High School, Dedham, Mass. Probs. I. and II.
CHARLES B. BOUTELLE, Waterville Academy, Maine. Prob. III.
M. H. DOOLITTLE, Sophomore Class, Antioch College, Yellow Springs, Ohio. Probs. III., IV., and first part of V.
W. D. MERSHON, Sophomore Class, Princeton College, N. J. Prob. IV.
R. E. LEONARD, Ward School, No. 32, New York City. Probs. I. and II.
J. B. FOSSETT, New London Institute, Ct. Probs. I., II., and III.
SAMUEL I. BALDWIN, Chester Institute, Chester, N. J. Probs. I. and II.
S. W. BURNHAM, New York City. Probs. I. and II.
M. L. STREATOR, Mayslick Academy, Ky. Probs. I., II., and III.
G. H. TOWER, Classical High School, Petersham, Mass. Probs. I. and II.
PERRIN B. PAGE, Literary Institute, Nunda, N. Y. Probs. I. and II.
J. D. VAN BUREN, Rensselaer Polytechnic Institute, Troy, N. Y. Probs. III., IV., and V.
O. B. WHEELER, Sophomore Class, University of Michigan, Ann Arbor. Probs. III., IV., and V.
JOSEPH COCKE and AYLETT B. COLEMAN, Locust Grove Academy, Albemarle Co., Va. Each Probs. I. and II.
M. K. BOSWORTH, Sophomore Class, Marietta College, Ohio. Probs. III., IV., and V.
W. A. AKEN, New Wilmington, Pa. Probs. I. and II.

The following solutions, unfortunately, did not reach us in time.

WILSON BERRYMAN and OTHO E. MICHAELIS, Sophomore Class, N. Y. Free Academy, each answered all the questions.

H. TIEMAN, Baltimore, Md., answered all the questions but the second part of Prob. V.

PHILO HOLCOMB, Hughes High School, Cincinnati, answered Prob. IV.

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